To prove: amortized cost to splay a tree with root t at node x is at most

$$3(r(t) - r(x)) + 1 = O\left(\lg \frac{s(t)}{s(x)}\right)$$

① If there are no rotations, i.e. x is the same as the root then $3(0) + 1 = 1 =$ Actual cost.

② Suppose there is at least one rotation $(x \neq t)$

CASE 1: Zig step (Actual cost $= 1$)

$$\text{amortized cost} = 1 + [r_1(x) + r_1(y) - r_0(x) - r_0(y)]$$

Only the rank of x and y can change.

$$\leq 1 + [r_1(x) - r_0(x)]$$

Since $r_0(y) \geq r_1(y)$
y parent of x

$$\leq 1 + 3(r_1(x) - r_0(x))$$

CASE 2: Zig Zig step

$$\text{amortized cost} = 2 + [\cancel{r_1(x)} + r_1(y) + r_1(z) - r_0(x) - r_0(y) - \cancel{r_0(z)}]$$

$$= 2 + r_1(y) + r_1(z) - r_0(x) - r_0(y)$$

$r_1(x) \geq r_1(y)$, $r_0(x) \leq r_0(y)$

$$\leq 2 + r_1(x) + r_1(z) - 2r_0(x)$$

To show that:

$$2 + r_1(x) + r_1(z) - 2r_0(x) \leq 3(r_1(x) - r_0(x))$$

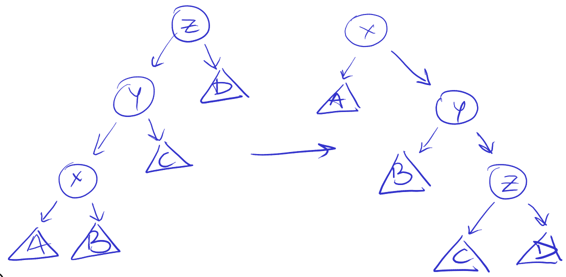
$$\equiv \underline{r_1(z) + r_0(x) - 2r_1(x)} \leq -2$$

We have: $r_1(z) + r_0(x) - 2r_1(x) \le -2$

$$= (r_1(z) - r_1(x)) + (r_0(x) - r_1(x))$$

$$= \lg\left(\frac{S_1(z)}{S_1(x)}\right) + \lg\left(\frac{S_0(x)}{S_1(x)}\right)$$

$$= \lg\left(\frac{S_1(z)}{S_1(x)} \cdot \frac{S_0(x)}{S_1(x)}\right) \longrightarrow (1)$$



but $S_1(x) \ge S_1(z) + S_0(x)$

$[S_1(x) = S_1(z) + S_0(x) + w(y)]$

$$\therefore \frac{S_1(z)}{S_1(x)} + \frac{S_0(x)}{S_1(x)} \le 1$$

Since $a+b \propto a*b$ for positive numbers.

$a + b \le 1$

$\Rightarrow a * b \le \frac{1}{4}$

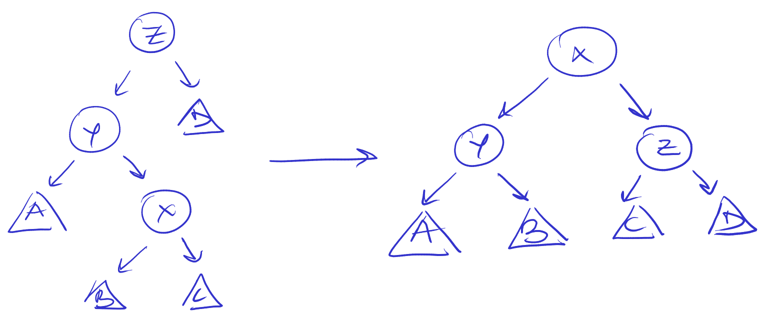
From this we get that $\frac{S_1(z)}{S_1(x)} * \frac{S_0(x)}{S_1(x)} \le \frac{1}{4} \to (2)$

Therefore, combining (1) and (2) we get that $r_1(z) + r_0(x) - 2r_1(x) \le \lg(1/4)$

$$\le -2$$

Therefore, amortized cost $\le 3(r_1(x) - r_0(x))$

CASE 3: (Zig Zag Step)



amortized cost =

$2 + \cancel{r_1(x)} + r_1(y) + r_1(z) - r_0(x) - r_0(y) - \cancel{r_0(z)}$

$= 2 + r_1(y) + r_1(z) - r_0(x) - r_0(y)$

$\le 2 + r_1(y) + r_1(z) - 2r_0(x)$ $\qquad r_0(x) \le r_0(y)$

Now to show $2 + r_1(y) + r_1(z) - 2r_0(x) \le 3(r_1(x) - r_0(x))$

$\equiv r_1(y) + r_1(z) + r_0(x) - 3r_1(x) \le -2$

$$= \lg \left[\frac{S_1(y)}{S_1(x)} * \frac{S_1(z)}{S_1(x)} \right] + \underbrace{\boxed{r_0(x) - r_1(x)}}_{\leq 0}$$

can ignore for upper bound.

$$\leq \lg [\ \ldots\ ]$$

Same reasons as CASE 2.

$$\leq \lg(1/4) = -2$$

$\therefore$ Amortized cost $\leq 3(r_1(x) - r_0(x))$

Summary Amortized cost:

1. Zig step $\leq 3(r_1(x) - r_0(x)) + 1$
2. Zig-Zag Step
3. Zig-Zig Step

$\left.\right\} \leq 3(r_1(x) - r_0(x))$

Suppose a splay operation had k steps, only the last step could be a zig step since y would have to be the root of the tree.

Amortized cost of the whole splay operation is:

$$\underbrace{3r_k(x) - 3r_{k-1}(x) + 1}_{\text{upper bound for last step}} + \sum_{j=1}^{k-1} 3(r_j(x) - r_{j-1}(x))$$

$$= 3r_k(x) - 3r_{k-1}(x) + 1 + 3r_{k-1}(x) - 3r_0(x)$$

$$= 3r_k(x) - 3r_0(x) + 1.$$

Access Lemma Proved.